On the limiting procedure by which $SDiff(T^2)$ and $SU(\infty)$ are associated

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ABSTRACT

There have been various attempts to identify groups of area-preserving diffeomorphisms of 2-dimensional manifolds with limits of $SU(N)$ as $N \to \infty$. We discuss the particularly simple case where the manifold concerned is the two-dimensional torus T^2 and argue that the limit, even in the basis commonly used, is ill-behaved and that the large-N limit of $SU(N)$ is much larger than $SDiff(T^2)$.

I. INTRODUCTION

Groups of area-preserving diffeomorphisms and their Lie algebras have recently been the focus of much attention in the physics literature. Hoppe [1] has shown that in a suitable basis, the Lie algebra of the group $SDiff(S^2)$ of area-preserving diffeomorphisms of a sphere tends to that of $SU(N)$ as $N \to \infty$. Similar arguments have been made associating various infinite limits of Lie algebras of classical groups with Lie algebras of groups of area-preserving

diffeomorphisms of 2-dimensional surfaces. This has obvious interest in connection with gauge theories of $SU(N)$ for large N. The use of $SU(N)$ for finite N as an approximation to groups of area-preserving diffeomorphisms has also been used in studies of supermembranes [2–4] and in particular has been used to argue for their instability. The authors of references [3] and [4] have especially emphasized the difficulties in relating such infinite limits with Lie algebras of area-preserving diffeomorphisms. Various authors have considered special limits and/or large-N limits of other classical Lie algebras [6–10] as relevant for 2-manifolds other than spheres. The purpose of this Letter is to clarify the nature of the limiting procedure by which $SU(\infty)$ has been related to $SDiff(T^2)$.

II. THE LIE ALGEBRAS OF $SDIFF(T^2)$

We follow here the treatment of [7], which is particularly clear. The torus T^2 is represented by the plane $\mathbb{R}^2$ with coordinates x and y and the identifications

$$(x, y) = (x + 2\pi, y) \tag{1}$$

and

$$(x, y) = (x, y + 2\pi) \tag{2}$$

A basis for functions on the torus is chosen as

$$Y_{mn}(x, y) = exp\left[i(mx + ny)\right] \tag{3}$$

with m, n running over all integers. The local area-preserving diffeomorphisms are then generated by the vector fields

$$L_{mn} = (\epsilon^{ab}\partial_b Y_{mn})\partial_a = iexp\left[i(mx+ny)\right](n\partial_x - m\partial_y) \tag{4}$$

with indices $a, b = 1, 2$. In other words, the divergence-free vector fields are those which are the curl of something else.

The generators clearly close under commutation, with the commutator

$$[L_{mn}, L_{m',n'}] = (mn' - m'n)L_{m+m',n+n'} \tag{5}$$

III. THE LIE ALGEBRA OF $SU(N)$

To construct the Lie algebra of $SU(N)$, again following [7], we sketch the basic idea. Fix a positive integer N and a complex number ω such that $\omega^N = 1$ but $\omega^r \neq 1$ for $0 < r < N$. ω is called a primitive root of unity. Then we have $\omega = exp(2\pi ik/N)$ for some k relatively prime to N. Now we find unitary, traceless matrices g and h such that

$$hg = \omega gh \tag{6}$$

Then the set of matrices

$$J_{m,n} = \omega^{mn/2} g^m h^n \tag{7}$$

for $0 \leq m, n < N$ are linearly independent and are a basis for the $N \times N$ matrices. $J_{0,0} = 1$, and all the other $J_{m,n}$ are traceless and satisfy $J^\dagger_{m,n} = J_{-m,-n}$. Leaving out $J_{0,0}$, the scaled matrices $J'_{m,n} = iN/(2k\pi)J_{m,n}$ generate $SU(N)$ with the commutation relations

$$\left[J'_{m,n}, J'_{m',n'}\right] = \frac{N}{k\pi}\sin\left(\frac{k\pi}{N}(mn' - m'n)\right)J'_{m+m',n+n'} \tag{8}$$

IV. THE $N \to \infty$ LIMIT

The claim now is that in the limit $N \to \infty$ that the commutation relations in equation III go over to those in equation II. Naively, of course, one would like to argue that as $N \to \infty$,

$$\frac{N}{k\pi} \sin\left(\frac{k\pi}{N}(mn' - m'n)\right) = (mn' - m'n) + O(1/N^2) \tag{9}$$

and drop the terms of order $1/N^2$ and higher. However, let us keep the next term and examine whether or not it can indeed be taken to be small.

$$\frac{N}{k\pi} \sin\left(\frac{k\pi}{N}(mn' - m'n)\right) = (mn' - m'n) - \frac{1}{3!}\frac{(k\pi)^2}{N^2}(mn' - m'n)^3 + \ldots \tag{10}$$

Now consider any choice of $(m, n) = (N/a, 0)$ and $(m', n') = (0, N/b)$ where a and b are arbitrary integers that divide N (including one). Then

$$\frac{(k\pi)^2}{N^2}(mn' - m'n)^3 = \frac{(k\pi)^2}{a^3b^3}N^4 \tag{11}$$

which is clearly *not* negligible as $N \to \infty$. It would seem that there are many elements of the Lie algebra of $SU(N)$ which do not belong to $SDiff(T^2)$.

This is in keeping with ideas raised in [11] suggesting that $SU(\infty)$ is much larger than the group of area-preserving diffeomorphisms of a surface, and perhaps descibes some sort of theory incluing topology change. Other work demonstrating that topologically, $SDiff(T^2)$, and indeed all the area-preserving diffeomorphism groups, are inequivalent to $SU(\infty)$ is in [12].

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